Mr Werke

PART I: Short Answer

Write the most simplified answer on the space provided. *(1.5 pts for each blank space)*

1. The domain of $f(x, y) = \sqrt{x^2 + y^2 - 1} + \ln(4 - x^2 - y^2)$ is $\{(x,y): 1 \leq x^2+y^2 < 4\}$

2. Evaluate (if exists)

a) $\lim_{(x,y)\to(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}} =$ 0

b) $\lim_{(x,y)\to(0,0)} \frac{x^3+\sin^2(y)}{2x^2+y^2} =$ $\nexists$

3. The total resistance R produced by three conductors with resistances R_1, R_2 & R_3 connected in parallel electrical circuit is given by $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$ then $\frac{\partial R}{\partial R_1} =$ $(R/R_1)^2 = \left(\frac{R_2R_3}{R_2R_3+R_1R_3+R_1R_2}\right)^2$

4. Let $f(x) = \begin{cases} \frac{x^2y - xy^3}{x^2+y^2} & if\,(x,y) \neq (0,0) \\ 0 & if\,(x,y) = (0,0) \end{cases}$ then $f_{xy}(0,0) =$ -1

5. At what points in a plane does $f(x,y) = \sin^{-1}(x^2 + y^2)$ continuous? $\{(x,y): x^2+y^2 \leq 1\}$

6. If $z = f(x, y)$,where f is differentiable, $x = g(t), y = h(t), g(3) = 2, g'(3) = 5, h(3) = 7, h'(3) = -4$

$f_x(2,7) = 6, and\ f_y(2,7) = -8$ then $\frac{dz}{dt}$ when $t = 3$ is 62

7. Let $z = x^2y + 3y^2$, $x = 2u$, $y = u + v$. Then $\frac{\partial z}{\partial u} =$ $12u^2+8uv+6u+6v = x^2+4xy+6y$

8. Consider the function $f(x,y,z) = x\sin(yz)$ then find

a) The directional derivative of f at (1,3,0) in the direction of $v = i+2j+k$ is $3/\sqrt{6} = \frac{\sqrt{6}}{2}$

b) The direction in which f *increases most rapidly* at (1, 3, 0) is $3k$ or $(0,0,3)$

9. The *tangent plane* to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1) is $4x+2y-z=3$
$z = 3+4(x-1)+2(y-1)$ or $z = 4x+2y-3$

10. Evaluate

a) $\int_0^1\int_1^2 4xy^2\,dxdy =$ 2

b) $\int_0^8\int_{\sqrt[3]{y}}^2 e^{x^4}\,dxdy =$ $\frac{1}{4}(e^{16}-1)$

11. Evaluate $\iint_R e^{-(x^2+y^2)}dA$ where $R = \{(x,y)/0 \leq x^2 + y^2 \leq 4, y \geq 0\}$ $-\frac{\pi}{2}(e^{-4}-1)$
$= \frac{\pi(e^4-1)}{2e^4}$

PART II: Work Out Problems

Show all the necessary steps and formulas clearly and completely.

1. Let $f(x,y)=4y^3+x^2-12y^2-36y+2$, Find **(5 pts)**

   a) All ***critical*** points of f.

   b) Identify the point(s) at which f has ***relative extreme*** values and at which it has ***saddle*** point(s) if any.

Sol^n Here $f(x,y)=4y^3+x^2-12y^2-36y+2$

$\Rightarrow f_x(x,y)=2x$ & $f_y(x,y)=12y^2-24y-36$ ✓ 0.5

To find the c.pts,

let $f_x(x,y)=0 \Rightarrow 2x=0 \Rightarrow x=0$

and $f_y(x,y)=0 \Rightarrow 12y^2-24y-36=0$ 0.5

$\Rightarrow y^2-2y-3=0$

$\Rightarrow (y-3)(y+1)=0$

$\Rightarrow y=3 \;/\; y=-1$

So $(0,-1)$ & $(0,3)$ are the critical pts ✓ 1

consider $(x_0,y_0)=(0,-1)$

$f_{xx}(x,y)=2$ so $f_{xx}(0,-1)=2$

$f_{yy}(x,y)=24y-24$ so $f_{yy}(0,-1)=-48$ 0.5

$f_{xy}(x,y)=0$

$$D(0,-1)=\begin{vmatrix} f_{xx}(0,-1) & f_{xy}(0,-1) \\ f_{xy}(0,-1) & f_{yy}(0,-1) \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -48 \end{vmatrix} = -96<0$$

$=2(-48)-0^2$

$\therefore (0,-1)$ is a saddle point ✓ 1

Consider $(x_0,y_0)=(0,3)$

$f_{xx}(0,3)=2$

$f_{yy}(0,3)=48$

$f_{xy}(0,3)=0$ 0.5

$$D=\begin{vmatrix} 2 & 0 \\ 0 & 48 \end{vmatrix}=96>0 \text{ and } f_{xx}(0,3)=2>0$$

$2(48)-0^2=$

$\therefore$ ~~(0,3) is the min~~ 1

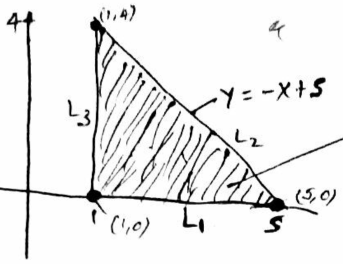
$f(x,y)$ has min pt at $(0,3)$

i.e $f(0,3)=-106$ is the min value //

2. Find the absolute extreme values of $f(x,y)=3+xy-x-2y$ on the set R which is a closed triangular region with vertices $(1,0),(5,0)$ & $(1,4)$

(5pts)

Sol^n:



* First we need to find at the critical point

$f_x = y-1 \Rightarrow f_x=0$ for $y=1$

$f_y = x-2 \Rightarrow f_y=0$ for $x=2$ (1)

$\Rightarrow (2,1)$ is the only cr. pt

~~$f_{xx}=0$, $f_{yy}=0$ and $f_{xy}=1$~~ $f(2,1)=1$ (0.5)

~~$D(2,1)=\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} = -1<0$~~

~~$\Rightarrow (2,1)$ is the saddle point.~~

* Now we check on the boundary

(0.5)

* on the line L_1: $y=0$, then $f(x,0)=3-x$ for $1\le x\le 5$. Thus -2 is min and 2 is max

* on the Line L_2, $y=-x+5$, then $f(x,-x+5)=g(x)=-x^2+6x-7$ for $1\le x\le 5$. Thus the min is -2 at $x=1, x=5$ and the max is 2 at $x=3$. (1) (0.5)

* On L_3: $x=1$, then $f(1,y)=3+y-1-2y=2-y$ for $0\le y\le 4$. Thus 2 max and -2 min values

∴ The absolute max value is **2** (0.5). at $(1,0),(3,2)$, ~~$(1,0)$~~

The absolute min value is **-2** (0.5) $(5,0),(1,4)$

		min	max
l_1	$y=0$, $1\le x\le 5$ $f(x)=3-x$	-2	2
l_2	$y=-x+5$ $f(x)=-x^2+6x-7$	-2	2
l_3	$x=1$, $0\le y\le 4$ $f(y)=2-y$	-2	2
	Absolute	-2 at	2

3. The temperature of a point (x, y) on a circle is given by $T(x, y) = 1 + xy$; find the points of maximum and minimum temperatures on the circle $x^2 + y^2 = 1$. **(5pts)**

Sol$^{\underline{n}}$: Let $g(x,y) = x^2 + y^2 - 1$, since $T(x,y) = 1 + xy$

By Lagrange's Theorem

$$\Rightarrow \boxed{\begin{cases} x^2+y^2 = 1 \\ T_x = \lambda g_x \\ T_y = \lambda g_y \end{cases}} \quad \text{or} \quad \begin{cases} x^2+y^2 = 1 \\ y = \lambda 2x \\ x = \lambda 2y \end{cases} \quad \text{or} \quad \begin{cases} x^2+y^2 = 1 \\ \lambda = \frac{y}{2x} \\ \lambda = \frac{x}{2y} \end{cases} \quad (1)$$

$\Rightarrow \frac{y}{2x} = \frac{x}{2y} \Rightarrow 2y^2 = 2x^2 \Rightarrow y^2 - x^2 = 0 \Rightarrow (y-x)(y+x) = 0$

$\Rightarrow y = x$ or $y = -x$ (0.5)

If $y = x$, then $x^2 + x^2 = 1 \Rightarrow x^2 = \frac{1}{2} \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

$y = -x$, then $x^2 + x^2 = 1 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$ (0.5)

$\therefore$ The points are $\boxed{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})}$. (1)

Thus, $T(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1 + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1 + \frac{1}{2} = \frac{3}{2}$

$T(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1 + (-\frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}}) = 1 + \frac{1}{2} = \frac{3}{2}$

$T(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}) = 1 + (\frac{1}{\sqrt{2}})(-\frac{1}{\sqrt{2}}) = 1 - \frac{1}{2} = \frac{1}{2}$ (1)

$T(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}) = 1 + (-\frac{1}{\sqrt{2}})(\frac{1}{\sqrt{2}}) = 1 - \frac{1}{2} = \frac{1}{2}$

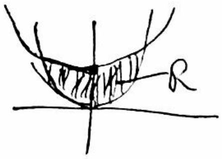
$\therefore \frac{3}{2}$ is the maximum temperature on $(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$. (0.5)

$\frac{1}{2}$ is the minimum temperature on $(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$. (0.5)

4. Evaluate the integral $\iint_R (x+2y)\,dA$, where R is the region bounded by $y=2x^2$ and $y=x^2+1$

(4pts) Sol: since $y=2x^2$ and $y=x^2+1$

$\Rightarrow 2x^2 = x^2+1 \Rightarrow x^2-1=0 \Rightarrow x=\pm 1$ 0.5



$\therefore \iint_R (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{x^2+1} (x+2y)\,dy\,dx = \int_{-1}^{1} (xy+y^2)\Big|_{2x^2}^{x^2+1} dx$ 1

$= \int_{-1}^{1} [(x(x^2+1)+(x^2+1)^2) - (x x^2 2 + 4x^4)]\,dx$

$= \int_{-1}^{1} [(\cancel{x^3}+x+\cancel{x^4}+2x^2+1 - \cancel{x^3} - \cancel{x^4}]\,dx$ $-8x^3-4x^4$ 1

$= \int_{-1}^{1} (-3x^4 - x^3 + 2x^2 + x + 1)\,dx$

$\cancel{= \int_{-1}^{1} (2x^2+x+1)dx}$

$= \left(\frac{-3}{5}x^5 - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x\right)\Big|_{-1}^{1}$

$= \left(-\frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1\right) - \left(\frac{3}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 1\right)$ 1

$= \frac{4}{3} + 2$

$= \frac{10}{3}$ 0.5

$= \frac{32}{15}$

$-\frac{6}{5} + \frac{4}{3} + 2$

$= \frac{-18+20+30}{15}$

$= \frac{32}{15}$